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C. D. Levermore
G. C. Pomraning
D. L. Sanzo
J. Wong

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by

C. D. Levermore*

G. C. Pomraning**

D. L. Sanzo**

J. Wong*

*Lawrence Livermore National Laboratory
Livermore, California 94550

**School of Engineering and Applied Science
University of California, Los Angeles
Los Angeles, California 90024

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Send proofs to:
G. C. Pomraning
6266 Boelter Hall
UCLA
Los Angeles, CA 90024

ABSTRACT

The time-independent linear transport problem in a purely absorbing (no scattering) random medium is considered. A formally exact equation for the ensemble averaged distribution function $\langle\Psi\rangle$ is derived. Under the assumption of a two-fluid statistical mixture, with the transition from one fluid to the other assumed to be determined by a Markov process, an exact solution to this equation for $\langle\Psi\rangle$ is obtained. In the source-free case, this solution is shown to agree with the result obtained by ensemble averaging simple exponential attenuation. Several approximations to the exact equation for $\langle\Psi\rangle$ are considered, and numerical results given to assess the accuracy of these approximations.

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I. INTRODUCTION

In this paper we consider the problem of describing particle transport in a statistical (random) medium. Specifically, we consider time-independent transport in a medium in which the only particle-medium interaction is annihilation (absorption). Allowing for an external source of particles in the medium, we then have the generic linear kinetic (transport) equation along a direction s given by

$$\frac{d\Psi(s)}{ds} + \sigma(s)\Psi(s) = S(s) \quad . \tag{1}$$

Here s is the spatial variable; $\Psi(s)$ is a distribution function defined such that the number of particles of speed v between s and s+ds is given by $v^{-1}\Psi(s)ds$; $\sigma(s)$ is the annihilation (absorption) cross section, defined such that $\sigma(s)ds$ is the probability of absorption for a particle traversing a distance ds; and S(s) is the external source strength, defined such that S(s)ds is the number of particles introduced into the medium per unit time between s and s+ds. If Eq. (1) is assumed to hold for $s > 0$, we then have the boundary condition

$$\Psi(0) = \Psi_o \quad , \tag{2}$$

where Ψ_o is the prescribed incident distribution at s = 0. We assume that σ and S in Eq. (1) are known only in some statistical or probabilistic sense. That is, at each space point s there is some time-independent probability that each of these two quantities will assume certain values. Accordingly, we consider σ and S, as well as the distribution function Ψ, to be random variables. Assuming we know the complete statistical description of σ and S, we seek the solution for $\langle\Psi\rangle$, the ensemble averaged (expected value) of the distribution function Ψ. We emphasize that Eq. (1) is a transport equation for particle propagation along a particular direction s in a three-dimensional setting. That is, our analysis and results are applicable to a random, three-dimensional medium.

There are conceptually two distinct ways to proceed. In the first approach, one can immediately write the solution to Eqs. (1) and (2) as

$$\Psi(s) = \Psi_o \exp\left[-\int_o^s ds'\sigma(s')\right] + \int_o^s ds'S(s')\exp\left[-\int_{s'}^s ds''\sigma(s'')\right] \quad , \quad (3)$$

and then ensemble average the right-hand side of Eq. (3) to obtain $\langle\Psi(s)\rangle$. Alternately, as a second approach one could develop, from Eqs. (1) and (2), a transport-like equation for $\langle\Psi(s)\rangle$, and then solve this equation to obtain the ensemble averaged solution. In this paper, we primarily focus on the second approach, but we also consider the first approach in the source-free (S=0) case.

Specifically, in Sec. II we develop the details of the second approach by using a projection operator technique, the method of smoothing as described by Keller[1,2,3] and Frisch[4], to derive a formally exact equation for $\langle\Psi(s)\rangle$. This equation contains an infinite series, with the nth term in this series involving an n-fold integral arising from n applications of the inverse transport operator. This multiple integral acts on various spatial correlation functions describing the statistical nature of the medium. For small statistical fluctuations, this infinite series can be truncated to a single term to obtain the lowest order (in the smallness parameter describing the fluctuations) statistical correction. The integral operator in this lowest order approximation can be localized by invoking a standard Fokker-Planck approximation.

These formal results are specialized, in Sec. III, to a statistical mixture of two immiscible fluids, with σ and S at any space point each taking one of the two values, that associated with each fluid. We show that the assumption of a Markov (Poisson) process for the transition from one fluid to the other allows an explicit calculation of all of the required spatial correlations. In addition, under this Markovian model, one can also obtain an analytic expression for the probability density distribution function corresponding to the optical depth random variable τ, defined as

$$\tau(s) = \int_0^s ds'\sigma(s') \quad . \tag{4}$$

This distribution function can be used to ensemble average the right-hand side of Eq. (3) in the source-free (S=0) case, since in this case Eq. (3) simply becomes $\Psi(s) = \Psi_0 \exp(-\tau)$.

In Sec. IV we show that the transport-like equation for $\langle\Psi(s)\rangle$ derived in Sec. II can be solved exactly for the two-fluid Markovian medium, in the special case of a homogeneous (spatially independent statistics for σ and S) medium. We numerically compare this exact solution with the result predicted by the small fluctuation equation (that which truncates the infinite series to a single term), as well as to the Fokker-Planck approximation to this small fluctuation equation. We also use the probability density distribution function for the optical depth τ obtained in Sec. III to carry out the details of ensemble averaging Eq. (3) in the source-free (S=0) case. The results of these two approaches to obtain $\langle\Psi(s)\rangle$ in a source-free medium are shown to be identical. The final section of the paper is devoted to a few concluding remarks.

As is clear from the above outline, the emphasis in this paper is on particle transport in a random medium composed of two immiscible turbulently mixed materials. This work was motivated by the need for an accurate transport description in the calculation of the performance of laser- or beam-driven fusion pellets. At an interface between two materials, these pellets are susceptible to Rayleigh-Taylor instabilities which can lead to a two-fluid turbulent mixture around the interface. A review of Rayleigh-Taylor instabilities within the context of inertially confined fusion has recently been given by Jacobs.[5] We intend to implement our formalism in the laser fusion code LASNEX used at the Lawrence Livermore National Laboratory. Other areas of application also come to mind. In a boiling-water nuclear reactor, the water, which acts as both coolant and moderator, is in a two-fluid random state (liquid and vapor). A proper treatment of the neutron transport must take the statistical nature of the mixture into account. In shielding calculations through concrete, the random nature of the materials (e.g., gravel) in the concrete implies a need for a statistical transport treatment to obtain an accurate measure of the shield effectiveness. Still another area of application is the calculation of light transport through a two-component random medium, such as sooty air or murky water. In general, there seems to be numerous areas of application for a transport theory for random media.

Finally, we note that the equation of radiative transfer with certain stochastic coefficients has been studied rather extensively with the astrophysical community.[6,7] However, the emphasis in this work has been on line transport with random Doppler shifts of the absorption coefficient due to small random velocity fields. The problem we treat, that of two turbulently mixed materials, is quite different from this astrophysical problem, even though both involve a stochastic linear transport equation.

II. THE EQUATION FOR $\langle\Psi(s)\rangle$

We rewrite Eq. (1) as

$$L\Psi + M\Psi = \langle S\rangle + q \quad , \tag{5}$$

where $\langle S\rangle$ is the ensemble averaged source, L is the ensemble averaged transport operator given by

$$L = \frac{d}{ds} + \langle\sigma\rangle \quad , \tag{6}$$

and q and M are the corresponding fluctuating quantities, i.e.,

$$q = S - \langle S\rangle \quad ; \qquad M = \sigma - \langle\sigma\rangle \quad . \tag{7}$$

We now introduce $\phi(s)$ as the fluctuating portion of $\Psi(s)$, i.e.,

$$\Psi = \langle\Psi\rangle + \phi \quad . \tag{8}$$

We note that q, M, and ϕ all have a zero expected value, i.e.,

$$\langle q\rangle = \langle M\rangle = \langle\phi\rangle = 0 \quad . \tag{9}$$

Following Keller [1,2,3] and Frisch,[4] we use Eq. (8) in Eq. (5) and ensemble average to obtain

$$L\langle\Psi\rangle + \langle M\phi\rangle = \langle S\rangle \quad . \tag{10}$$

The term $\langle M\phi\rangle$ in Eq. (10) represents the statistical correction to the transport description. To compute this quantity, we subtract Eq. (10) from Eq. (5) to obtain

$$L\phi = q - M\langle\Psi\rangle + [\langle M\phi\rangle - M\phi] \quad , \tag{11}$$

or

$$\phi = L^{-1}(q - M\langle\Psi\rangle) + (B_1 - B_2)\phi \quad . \tag{12}$$

Here B_1 is the projection operator defined by

$$B_1\phi = L^{-1}\langle M\phi\rangle = \langle L^{-1}M\phi\rangle \quad , \tag{13}$$

B_2 is the corresponding unprojected operator defined by

$$B_2\phi = L^{-1}M\phi \quad , \tag{14}$$

and the inverse operator L^{-1} is explicitly given by (since ϕ vanishes at $s = 0$)

$$L^{-1}\phi(s) = \int_o^s ds'\phi(s') \exp\left[-\int_{s'}^s ds''\langle\sigma(s'')\rangle\right] \quad . \tag{15}$$

We rewrite Eq. (12) as

$$(I - B_1 + B_2)\phi = L^{-1}(q - M\langle\Psi\rangle) \quad , \tag{16}$$

which has the formal Neumann series solution

$$\phi = \sum_{n=0}^{\infty} (-1)^n (B_2 - B_1)^n L^{-1}(q - M\langle\Psi\rangle) \quad . \tag{17}$$

Operating on Eq. (17) with the operator M and ensemble averaging gives

$$\langle M\phi\rangle = L \sum_{n=0}^{\infty} (-1)^n \left(\hat{T}_{n+2} - \tilde{T}_{n+2}\right) \quad , \tag{18}$$

where

$$\hat{T}_{n+2} = \langle B_2 (B_2 - B_1)^n L^{-1} q\rangle \qquad n \geq 0 \quad , \tag{19}$$

and

$$\tilde{T}_{n+2} = \langle B_2 (B_2 - B_1)^n B_2\rangle\langle\Psi\rangle \quad , \qquad n \geq 0 \quad . \tag{20}$$

Use of Eq. (18) in Eq. (10) gives

$$L\langle\Psi\rangle + L \sum_{n=0}^{\infty} (-1)^n \left(\hat{T}_{n+2} - \tilde{T}_{n+2}\right) = \langle S\rangle \quad . \tag{21}$$

Equation (21) is the formally exact transport-like equation for $\langle\Psi\rangle$, the ensemble averaged distribution function. The infinite series in this equation is the statistical correction to the transport description.

From their definitions according to Eqs. (19) and (20), one can easily deduce recurrence relationships for $\hat{T}_n$ and $\tilde{T}_n$ given by

$$\hat{T}_n = \langle (L^{-1}M)^{n-1} L^{-1} q\rangle - \sum_{i=2}^{n-2} \langle (L^{-1}M)^i\rangle \hat{T}_{n-i} \quad , \qquad n \geq 3 \quad , \tag{22}$$

$$\tilde{T}_n = \langle (L^{-1}M)^n \rangle \langle \Psi \rangle - \sum_{i=2}^{n-2} \langle (L^{-1}M)^i \rangle \tilde{T}_{n-i} \quad , \qquad n > 3 \quad . \tag{23}$$

These recurrence relationships are initiated by the explicit n=2 expressions

$$\hat{T}_2 = \langle L^{-1}ML^{-1}q \rangle \quad ; \qquad \tilde{T}_2 = \langle L^{-1}ML^{-1}M \rangle \langle \Psi \rangle \quad . \tag{24}$$

From these recurrence relationships one can prove, by induction, that $\hat{T}_n$ and $\tilde{T}_n$ can be written in explicit form as

$$\hat{T}_n = \sum_i a_i \langle (L^{-1}M)^{p_1} \rangle \langle (L^{-1}M)^{p_2} \rangle \ldots \langle (L^{-1}M)^{p_i} L^{-1}q \rangle \quad , \quad n > 2 \; , \tag{25}$$

$$\tilde{T}_n = \sum_i a_i \langle (L^{-1}M)^{p_1} \rangle \langle (L^{-1}M)^{p_2} \rangle \ldots \langle (L^{-1}M)^{p_i} \rangle \langle \Psi \rangle \quad , \quad n > 2 \; . \tag{26}$$

The powers p_k can assume any nonnegative integer values subject to the constraint

$$p_1 + p_2 + \ldots + p_i = m \quad , \tag{27}$$

where $m = n-1$ for Eq. (25) and $m = n$ for Eq. (26). The sum over i in Eqs. (25) and (26) is over all possible combinations for the powers p_k, and $a_i = +1$ for an odd number of terms in the product involving ensemble averaged operators, and $a_i = -1$ for an even number of terms. As an explicit example, we have

$$\hat{T}_5 = \langle (L^{-1}M)^4 L^{-1}q \rangle - \langle (L^{-1}M)^2 \rangle \langle (L^{-1}M)^2 L^{-1}q \rangle$$
$$- \langle (L^{-1}M)^3 \rangle \langle L^{-1}ML^{-1}q \rangle \tag{28}$$

$$\tilde{T}_5 = \langle(L^{-1}M)^5\rangle\langle\Psi\rangle - \langle(L^{-1}M)^2\rangle\langle(L^{-1}M)^3\rangle\langle\Psi\rangle$$

$$- \langle(L^{-1}M)^3\rangle\langle L^{-1}M)^2\rangle\langle\Psi\rangle \quad . \tag{29}$$

To proceed, we define the nth order spatial correlations according to

$$\hat{N}_n(s_1,\ldots,s_n) = \langle M(s_1)M(s_2) \,..\, M(s_{n-1})q(s_n)\rangle \quad , \tag{30}$$

and

$$\tilde{N}_n(s_1,\ldots s_n) = \langle M(s_1)M(s_2) \ldots M(s_{n-1})M(s_n)\rangle \quad , \tag{31}$$

and, in analogy to Eq. (4), we define τ_n as the optical depth corresponding to a distance s_n, i.e.,

$$\tau_n = \int_o^{s_n} ds'\sigma(s') \quad . \tag{32}$$

In terms of these definitions, we can write, using Eq. (15) for L^{-1},

$$L\hat{T}_n = \int_o^s ds_1 \int_o^{s_1} ds_2 \ldots \int_o^{s_{n-2}} ds_{n-1} \exp[-(\langle\tau\rangle-\langle\tau_{n-1}\rangle)]$$

$$\times \sum_i a_i[\tilde{N}_{p_1}\tilde{N}_{p_2} \ldots \tilde{N}_{p_{i-1}}\hat{N}_{p_i}] \quad , \tag{33}$$

$$L\tilde{T}_n = \int_o^s ds_1 \int_o^{s_1} ds_2 \ldots \int_o^{s_{n-2}} ds_{n-1} \exp[-(\langle\tau\rangle-\langle\tau_{n-1}\rangle)]$$

$$\times \sum_i a_i[\tilde{N}_{p_1}\tilde{N}_{p_2} \ldots \tilde{N}_{p_{i-1}}\tilde{N}_{p_i}]\langle\Psi(s_{n-1})\rangle \quad , \tag{34}$$

where the arguments in the product terms involving the $\hat{N}_{p_i}$ and $\tilde{N}_{p_k}$ are s, s_1, ... , s_{n-1} in this order. Again, as an explicit example we have

$$L\hat{T}_5 = \int_o^s ds_1 \int_o^{s_1} ds_2 \int_o^{s_2} ds_3 \int_o^{s_3} ds_4 \, \exp[-(\langle\tau\rangle-\langle\tau_4\rangle)]$$

$$\times [\hat{N}_5(s,s_1,s_2,s_3,s_4) - \tilde{N}_2(s,s_1)\hat{N}_3(s_2,s_3,s_4)$$

$$- \tilde{N}_3(s,s_1,s_2)\hat{N}_2(s_3,s_4)] \quad , \tag{35}$$

$$L\tilde{T}_5 = \int_o^s ds_1 \int_o^{s_1} ds_2 \int_o^{s_2} ds_3 \int_o^{s_3} ds_4 \, \exp[-(\langle\tau\rangle-\langle\tau_4\rangle)]$$

$$\times [\tilde{N}_5(s,s_1,s_2,s_3,s_4) - \tilde{N}_2(s,s_1)\tilde{N}_3(s_2,s_3,s_4)$$

$$- \tilde{N}_3(s,s_1,s_2)\tilde{N}_2(s_3,s_4)]\langle\Psi(s_4)\rangle \quad . \tag{36}$$

To proceed further, one needs to specify a statistical model to compute the spatial correlations $\hat{N}_n$ and $\tilde{N}_n$. We consider one such model in the next section.

To summarize our considerations thus far, Eq. (21) is the formally exact transport-like equation for the ensemble averaged distribution function $\langle\Psi(s)\rangle$, with $\hat{T}_n$ and $\tilde{T}_n$ given by Eqs. (33) and (34). The statistics of the medium enter through the multipoint spatial correlations $\hat{N}_n$ and $\tilde{N}_n$ defined by Eqs. (30) and (31), with q and M in these equations given by S - ⟨S⟩ and σ - ⟨σ⟩, respectively. We note that the statistical corrections in Eq. (21), embodied in the infinite series, involve nonlocal (multiple integral) operators. It is clear from Eqs. (19) and (20) that $\hat{T}_n$ and $\tilde{T}_n$ decrease geometrically with n in the smallness parameter characterizing the statistical fluctuations. Accordingly, one can obtain the lowest order, in this smallness parameter, approximation by keeping only the first term in the infinite series in Eq. (21). We then have, as the small fluctuation approximation, the transport-like equation

$$\frac{d\langle\Psi\rangle}{ds} + \langle\sigma\rangle\langle\Psi\rangle + \int_o^s ds_1 \, \exp[-(\langle\tau\rangle-\langle\tau_1\rangle)]$$

$$\times \, [\langle M(s)q(s_1)\rangle - \langle M(s)M(s_1)\rangle\langle\Psi(s_1)\rangle] = \langle S\rangle \quad . \tag{37}$$

We see that even in this lowest order approximation, the statistical correction in the transport-like equation involves an integral operator.

One can localize the integral operator in Eq. (37) by employing a Fokker-Planck approximation. Specifically, we approximate $\langle\Psi(s_1)\rangle$ in Eq. (37) by an Nth order Taylor series expansion about the point s, i.e.,

$$\langle\Psi(s_1)\rangle \approx \sum_{n=0}^{N} \frac{1}{n!} (s_1 - s)^n \frac{d^n\langle\Psi(s)\rangle}{ds^n} \quad . \tag{38}$$

Use of Eq. (38) in Eq. (37) and integrating term by term gives an Nth order Fokker-Planck approximation to the small fluctuation equation. To obtain explicit results, we consider the special case of a medium in which $\langle\sigma\rangle$ is a slowly varying (essentially constant) function of position. We then have

$$\langle\tau\rangle - \langle\tau_1\rangle = \langle\sigma\rangle(s-s_1) \quad . \tag{39}$$

We further assume that the two-point spatial correlations in Eq. (37) are exponential in form, i.e.,

$$\langle M(s)q(s_1)\rangle = \alpha \, \exp[-\eta|s-s_1|] \quad , \tag{40}$$

$$\langle M(s)M(s_1)\rangle = \beta \, \exp[-\eta|s-s_1|] \quad , \tag{41}$$

where α, β, and η are slowly varying (essentially constant) functions of position. In the next section we present a Markov statistical model for a two-fluid mixture which predicts two-point spatial correlations of precisely this form, and gives explicit expressions for α, β, and η in terms of the parameters in the Markov model. Using Eqs. (38) through (41) in Eq. (37) we find the Nth order Fokker-Planck approximation to the small fluctuation equation given by

$$\frac{d\langle\Psi\rangle}{ds} + \langle\sigma\rangle\langle\Psi\rangle - \frac{\beta}{\hat{\sigma}} \sum_{n=0}^{N} \left(- \frac{1}{\hat{\sigma}} \frac{d}{ds}\right)^{n} \langle\Psi\rangle = \langle S\rangle - \frac{\alpha}{\hat{\sigma}} \quad , \tag{42}$$

where we have defined

$$\hat{\sigma} = \langle\sigma\rangle + \eta \quad . \tag{43}$$

In obtaining Eq. (42) from Eq. (37), we have replaced the lower integration limit in Eq. (37) by $s_1 = -\infty$, which means we are neglecting terms of order $\exp(-\hat{\sigma}s)$. This is consistent with $\hat{\sigma}$ being large, which implies rapid convergence of the sum in Eq. (42). We note that the assumption that $\hat{\sigma}$ is large implies in general that η^{-1}, the spatial correlation length, is small. As we shall see in Sec. IV, the small fluctuation result, Eq. (37), and its Fokker-Planck approximation, Eq. (42), can yield nonphysical results if the fluctuations are, in fact, not small.

To summarize the results of this section, we have developed three descriptions of time-independent transport in a purely absorbing statistical medium. These are: (1) Eq. (21), which is exact but very formal; (2) Eq. (37) which assumes small fluctuations; and (3) Eq. (42) which assumes small fluctuations, exponential spatial correlations with a small correlation length, and slowly varying spatial properties $\langle\sigma\rangle$, α, β, and η. In the next section we present a Markov statistical model for a two-fluid mixture which yields explicit results for all of the required spatial correlations $\hat{N}_n$ and $\tilde{N}_n$. In particular, this model predicts two-point spatial correlations of the exponential form given by Eqs. (40) and (41).

III. A MARKOV STATISTICAL MODEL

We consider a static turbulent (random) mixture of two immiscible fluids which we denote by fluid A and fluid B. We associate a cross section σ_i and source S_i (i=A,B) with each fluid. As a particle travels through this fluid mixture, it will pass through alternating fluid packets of A and B. We assume that the statistics of this situation can be described by a stationary Markov process in the following sense. Given that a particle is in fluid A at position s, the probability of finding itself (in the absence of absorption) in fluid B at a position s + ds is simply given by ds/λ_A. Similarly, given that a particle is in fluid B at position s, the probability of finding itself (in the absence of absorption) in fluid A at a position s + ds is given by ds/λ_B. We take the σ_i, S_i, and λ_i to be constants, independent of position.

For this two state Markov chain, we define the transition probability function $P_{ij}(s,t)$, i = A,B, as

$$P_{ij}(s,t) = P[X(t) = j | X(s) = i] \quad , \qquad t > s \quad , \tag{44}$$

where the right-hand side of this equation is the conditional probability that the random variable X, which we define to be the state of the fluid, takes on the state j at a distance t (from some defined origin), given that the variable was in state i at a distance s from this origin. Without loss of generality, we assume $t > s$. Since the probability of transition from fluid i to fluid j in a distance dt is given by dt/λ_i, one can perform a transition balance into and out of a given state, as a function of t for a fixed s. These balance equations are well known as the Chapman-Kolmogorov equations (forward form),[8] and are given by

$$\partial P_{AB}/\partial t = - P_{AB}/\lambda_B + P_{AA}/\lambda_A \quad , \tag{45}$$

$$\partial P_{AA}/\partial t = - P_{AA}/\lambda_A + P_{AB}/\lambda_B \quad , \tag{46}$$

$$\partial P_{BA}/\partial t = - P_{BA}/\lambda_A + P_{BB}/\lambda_B \quad , \tag{47}$$

$$\partial P_{BB}/\partial t = - P_{BB}/\lambda_B + P_{BA}/\lambda_A \tag{48}$$

The boundary conditions on these differential equations are given by

$$P_{AA}(s,s) = P_{BB}(s,s) = 1 \quad , \tag{49}$$

$$P_{AB}(s,s) = P_{BA}(s,s) = 0 \quad . \tag{50}$$

It is clear that

$$P_{AB}(s,t) + P_{AA}(s,t) = 1 \quad , \tag{51}$$

$$P_{BA}(s,t) + P_{BB}(s,t) = 1 \quad , \tag{52}$$

and hence two of the equations in Eqs. (45) through (48) are redundant.

The solution of Eqs. (45) through (50) is

$$p_{AB}(s,t) = (\lambda_A + \lambda_B)^{-1}\ \lambda_B\left(1 - e^{-d/\lambda_p}\right) \quad , \tag{53}$$

$$p_{AA}(s,t) = (\lambda_A + \lambda_B)^{-1}\ \left(\lambda_A + \lambda_B\ e^{-d/\lambda_p}\right) \quad , \tag{54}$$

$$p_{BA}(s,t) = (\lambda_A + \lambda_B)^{-1}\ \lambda_A\left(1 - e^{-d/\lambda_p}\right) \quad , \tag{55}$$

$$p_{BB}(s,t) = (\lambda_A + \lambda_B)^{-1}\ \left(\lambda_B + \lambda_A\ e^{-d/\lambda_p}\right) \quad , \tag{56}$$

where we have defined

$$\lambda_p^{-1} = \lambda_A^{-1} + \lambda_B^{-1} \quad ; \qquad d = t - s \quad . \tag{57}$$

We note that these four conditional probabilities are independent of the choice of origin for the position variable; they depend only upon the distance $t - s$ between the points t and s.

We now define $p_i(s)$ as the probability that at any point s the fluid is in state i, i.e.,

$$p_i(s) = P[X(s) = i] \quad . \tag{58}$$

Since the λ_i have been assumed to be independent of position, it is clear that the p_i are also independent of position. That is, the s dependence on the left-hand side of Eq. (58) is redundant. In terms of the p_i, we have for the ensemble averaged cross section and source,

$$\langle\sigma\rangle = p_A\sigma_A + p_B\sigma_B \quad , \tag{59}$$

$$\langle S\rangle = p_A S_A + p_B S_B \quad . \tag{60}$$

The total probabilities p_i are related to the conditional probabilities P_{ij} by

$$p_A(t) = P_{AA}(s,t)p_A(s) + P_{BA}(s,t)p_B(s) \quad , \tag{61}$$

with a similar expression for $p_B(t)$ found by interchanging the indices A and B. Equation (61) holds in general, and in particular for our case in which the p_i are independent of position. Use of Eqs. (53) through (56) in Eq. (61) gives

$$p_A(t) = (\lambda_A + \lambda_B)^{-1} \{\lambda_A + [\lambda_B p_A(s) - \lambda_A p_B(s)]e^{-d/\lambda_p}\} \quad . \tag{62}$$

For $p_A(t)$ to be constant, independent of t, Eq. (62) implies

$$\lambda_B p_A - \lambda_A p_B = 0 \quad . \tag{63}$$

We then deduce

$$p_i = (\lambda_A + \lambda_B)^{-1}\lambda_i \quad , \tag{64}$$

as the relationship between the p_i and the λ_i. We shall shortly see that λ_i, and hence p_i, is proportional to the volume fraction of the ith fluid [see Eqs. (80) and (81)].

We now turn to the calculation of the two-point auto-correlation function for the cross section. We have

$$\langle M(s)M(t)\rangle = \langle[\sigma(s)-\langle\sigma\rangle][\sigma(t)-\langle\sigma\rangle]\rangle$$

$$= (\sigma_A-\langle\sigma\rangle)^2 P_{AA}(s,t)p_A + (\sigma_B-\langle\sigma\rangle)^2 P_{BB}(s,t)p_B$$

$$+ (\sigma_A-\langle\sigma\rangle)(\sigma_B-\langle\sigma\rangle)[P_{AB}(s,t)p_A + P_{BA}(s,t)p_B] \quad . \qquad (65)$$

This gives, using Eqs. (53) through (56) for the $P_{ij}(s,t)$,

$$\langle M(s)M(t)\rangle = (\sigma_A - \sigma_B)^2 p_A p_B \, e^{-d/\lambda_p} \quad , \qquad (66)$$

where we recall that d is the distance between the points t and s. A similar calculation for the two-point cross-correlation function between the cross section and the source gives

$$\langle M(s)q(t)\rangle = (\sigma_A - \sigma_B)(S_A - S_B)p_A p_B \, e^{-d/\lambda_p} \quad . \qquad (67)$$

In the notation of the last section [see Eqs. (40) and (41)], we then have

$$\alpha = (\sigma_A - \sigma_B)(S_A - S_B)p_A p_B \quad , \qquad (68)$$

$$\beta = (\sigma_A - \sigma_B)^2 p_A p_B \quad , \qquad (69)$$

$$\eta = \lambda_p^{-1} = \lambda_A^{-1} + \lambda_B^{-1} \quad , \qquad (70)$$

as the constants in the exponential two-point correlations. A similar exponential two-point auto-correlation as given by Eq. (66) was previously reported by Debye and Bueche[9] and Debye, Anderson, and Brumberger[10] within the context of light scattering by an inhomogeneous solid.

The higher order correlations, $\hat{N}_n$ and $\tilde{N}_n$ for $n > 2$ as defined by Eqs. (30) and (31), can be computed explicitly by an extension of these arguments. For a general n, these algebraic expressions are relatively complex. However, certain linear combinations of products of these higher order correlations are simple exponentials. Omitting the considerable algebraic detail, one finds

$$\sum_i a_i [\tilde{N}_{p_1} \tilde{N}_{p_2} \cdots \tilde{N}_{p_{i-1}} \hat{N}_{p_i}] = H_n \, e^{-(s_1 - s_n)/\lambda_p} \quad , \tag{71}$$

$$\sum_i a_i [\tilde{N}_{p_1} \tilde{N}_{p_2} \cdots \tilde{N}_{p_{i-1}} \tilde{N}_{p_i}] = K_n \, e^{-(s_1 - s_n)/\lambda_p} \quad , \tag{72}$$

where

$$H_n = (\sigma_A - \sigma_B)^{n-1}(S_A - S_B) p_A p_B (p_B - p_A)^{n-2} \quad , \qquad n > 2 \quad , \tag{73}$$

$$K_n = (\sigma_A - \sigma_B)^n p_A p_B (p_B - p_A)^{n-2} \quad , \qquad n > 2 \quad . \tag{74}$$

Here the spatial points are ordered such that $s_1 > s_2 > \ldots > s_n$, and the arguments in the product terms in Eqs. (71) and (72) involving the $\hat{N}_{p_i}$ and $\tilde{N}_{p_k}$ are $s_1, s_2, \ldots, s_n$ in this order. The coefficients a_i and the subscripts p_k are as discussed in the last section, just below Eq. (27).

The interesting point here is that the left-hand sides of Eqs. (71) and (72) are precisely of the form needed in the expressions for $\hat{T}_n$ and $\tilde{T}_n$ as introduced in the last section. That is, using Eqs. (71) and (72) in Eqs. (33) and (34) gives the relatively simple results for the nth order term in Eq. (21) as

$$L\hat{T}_n = \int_o^s ds_1 \int_o^{s_1} ds_2 \ldots \int_o^{s_{n-2}} ds_{n-1} H_n \exp[-\hat{\sigma}(s-s_{n-1})] \quad , \quad n > 2 \quad , \tag{75}$$

$$L\tilde{T}_n = \int_o^s ds_1 \int_o^{s_1} ds_2 \ldots \int_o^{s_{n-2}} ds_{n-1} K_n \langle\Psi(s_{n-1})\rangle \exp[-\hat{\sigma}(s-s_{n-1})] \ , \ n > 2 \ , \tag{76}$$

where [see Eqs. (43) and (70)]

$$\hat{\sigma} = \langle\sigma\rangle + \lambda_A^{-1} + \lambda_B^{-1} \quad . \tag{77}$$

In writing Eqs. (75) and (76) we have used an expression analogous to Eq. (39) for $\langle\tau\rangle - \langle\tau_{n-1}\rangle$ since our Markov statistical model is restricted to cases for which $\langle\sigma\rangle$ is independent of position. We note that H_n and K_n, and hence $\hat{T}_n$ and $\tilde{T}_n$, vanish for $n > 2$ in the special case that $p_A = p_B = 1/2$. Hence the small fluctuation approximation introduced in the last section, Eq. (37), is, in fact, exact for all size fluctuations when $p_A = p_B = 1/2$. For any other values of p_A and p_B, Eq. (37) is only strictly valid for vanishingly small fluctuations. As we shall see shortly, $p_A = p_B = 1/2$ implies equal volume fractions of the two fluids A and B.

Let us now address the question as to the physical meaning of our Markov model. Since the probability of transition from fluid i to fluid j in a distance ds is given by ds/λ_i, where λ_i is a constant, the distribution of chord lengths in a fluid packet is a classical Poisson process. That is, the chord length L of a given fluid packet is exponentially distributed, with a probability density function given by

$$f_i(L) = \lambda_i^{-1} e^{-L/\lambda_i} \quad , \qquad i = A,B \quad . \tag{78}$$

Equation (78) is implied by the Chapman-Kolmogorov equations by solving these equations after deleting the transition-in terms [the second term on the right-hand sides of Eqs. (45) through (48)]. The mean of this exponential distribution, $\langle L_i \rangle$, is given by

$$\langle L_i \rangle = \int_0^\infty dL L f_i(L) = \lambda_i \quad . \tag{79}$$

Thus the constant λ_i in the Markov model is just the average chord length through a fluid packet of type i. This average chord length is given by the Debye formula[9,10]

$$\lambda_i = 4V_i/S \quad , \tag{80}$$

where V_i is the volume associated with fluid i, and S is the common surface area between the fluid packets of fluids A and B. Using Eq. (80) in Eq. (64) we find

$$p_i = (V_A + V_B)^{-1} V_i \quad . \tag{81}$$

That is, the probability p_i is just the volume fraction of fluid i in the two-fluid stochastic mixture.

We now have the physical interpretation of our Markov model. The statistics of the two-fluid mixture are such that a particle traveling through this fluid sees alternating packets of fluids A and B, with the distance traveled (in the absence of absorption) in any fluid packet being a random variable with an exponential density distribution given by Eq. (78). Further, the parameter λ_i in this distribution is the average chord length through a fluid packet of type i, and is related to the volume fraction of fluid i in this two-fluid mixture through Eqs. (80) and (81). We note, however, that it is not sufficient to know the two volume fractions p_A and p_B. In addition to these volume fractions, one must know one of the λ_i to completely specify the statistics of the two-fluid mixture.

Before leaving this section, we use this Markov model to calculate another quantity which we shall find useful. The optical depth $\tau(s)$ between any two points a distance s apart, say s_o and s_o+s, is defined by

$$\tau(s) = \int_{s_o}^{s_o+s} ds'\sigma(s') \quad . \tag{82}$$

Since σ is a random variable, so is τ We seek the probability density function for the random variable τ, given a distance s. Since our medium is described by statistics which are independent of position, the point s_o is irrelevant; the random variable τ is independent of s_o. Since there are two states, A and B, between s_o and s_o+s, we have

$$\begin{aligned} \tau(s) = \sigma_A &\times [\text{total track length through A in distance s}] \\ &+ \sigma_B \times [\text{total track length through B in distance s}] \quad . \end{aligned} \tag{83}$$

To obtain the distributions of the total track length through A and B in a distance s, we make use of a problem outline given by Lindley.[11] Let the length a particle travels through in the ith packet of A, before finding itself in fluid B, be denoted by the random variable X_i. Similarly, let the length a particle travels through in the ith packet of B, before finding itself in fluid A, be denoted by the random variable Y_i. We know from

Eq. (78) that X_i and Y_i in our model are independent exponentially distributed random variables, and their cumulative distribution functions are given by

$$P(X_i < x) = G(x) = 1 - e^{-x/\lambda_A} \quad , \qquad 0 \leq x < \infty \quad , \tag{84}$$

$$P(Y_i < y) = H(y) = 1 - e^{-y/\lambda_B} \quad , \qquad 0 \leq y < \infty \quad . \tag{85}$$

We now define

$$G_n(x) = P\left(\sum_{i=1}^{n} X_i < x\right) \quad , \qquad n \geq 1 \quad , \tag{86}$$

$$H_n(y) = P\left(\sum_{i=1}^{n} Y_i < y\right) \quad , \qquad n \geq 1 \quad , \tag{87}$$

with

$$G_o(x) = H_o(y) = 1 \quad . \tag{88}$$

$G_n(x)$ and $H_n(y)$ represent, for $n \geq 1$, the distribution of the total track length in fluids A and B, respectively, in n packets of fluid.

We also define the random variable $\beta(s)$ as the total track length of fluid B in the distance s given that the point s_o is in fluid A. Similarly, we define the random variable $\alpha(s)$ as the total track length of fluid A in the distance s given that the point s_o is in fluid B. Then, according to Eq. (83) the optical depth as a function of the distance s, $\tau(s)$, is given by one of two expressions, namely

$$\tau(s) = \sigma_B\beta(s) + \sigma_A[s-\beta(s)] \quad , \qquad s_o \varepsilon A \quad , \tag{89}$$

or

$$\tau(s) = \sigma_A\alpha(s) + \sigma_B[s-\alpha(s)] \quad , \qquad s_o \varepsilon B \quad . \tag{90}$$

The cumulative distribution function $F(t,s)$ is the probability that $\tau(s)$ is less than a value t, given a geometric distance s. We have

$$F(t,s) = P[\tau(s) < t]$$
$$= P\{\sigma_B\beta(s) + \sigma_A[s-\beta(s)] < t\}P(s_o\varepsilon A)$$
$$+ P\{\sigma_A\alpha(s) + \sigma_B[s-\alpha(s)] < t\}(s_o\varepsilon B) \quad . \tag{91}$$

Recalling that

$$P(s_o\varepsilon A) = p_A \quad ; \qquad P(s_o\varepsilon B) = p_B \quad , \tag{92}$$

and rearranging Eq. (91), we find

$$F(t,s) = p_A P\left[\beta(s) > \frac{\sigma_A s - t}{\sigma_A - \sigma_B}\right] + p_B P\left[\alpha(s) < \frac{t - \sigma_B s}{\sigma_A - \sigma_B}\right] \quad . \tag{93}$$

If we label the fluids such that $\sigma_A > \sigma_B$, Eq. (93) immediately gives, since $0 < \alpha, \beta < s$,

$$F(t,s) = \begin{cases} 0 \quad , & \tau < \sigma_B s \\ 1 \quad , & \tau > \sigma_A s \end{cases} \tag{94}$$

which is just the physical statement that in a distance s the minimum optical depth is $\sigma_B s$ and the maximum optical depth is $\sigma_A s$.

To evaluate $F(t,s)$ for $\sigma_B s < t < \sigma_A s$, we need compute the distributions for $\alpha(s)$ and $\beta(s)$. To obtain the distribution for $\beta(s)$, we note that if there are exactly n transitions from state A to state B in a distance s-x, then the track length through the n packets of fluid B must lie between 0 and x. The probability of exactly n transitions from state A in a distance s-x is given by

$$P = G_n(s-x) - G_{n+1}(s-x) \quad . \tag{95}$$

Thus we may express the distribution for $\beta(s)$ as

$$P[\beta(s) < x] = \sum_{n=0}^{\infty} H_n(x)\left[G_n(s-x) - G_{n+1}(s-x)\right] \quad . \tag{96}$$

In a similar fashion, we can deduce the distribution for $\alpha(s)$ as

$$P[\alpha(s) < x] = \sum_{n=0}^{\infty} G_n(x)\left[H_n(s-x) - H_{n+1}(s-x)\right] \quad . \tag{97}$$

Now, it is known[12] that the sum of n identically distributed exponential random variables with parameter $1/\lambda$ is given by a gamma distribution with parameters n and $1/\lambda$. Thus we have in our case

$$G_n(x) = \int_o^x dx' \frac{(x'/\lambda_A)^{n-1} e^{-x'/\lambda_A}}{\lambda_A(n-1)!} \quad , \quad n \geq 1 \quad , \tag{98}$$

$$H_n(y) = \int_o^y dy' \frac{(y'/\lambda_B)^{n-1} e^{-y'/\lambda_B}}{\lambda_B(n-1)!} \quad , \quad n \geq 1 \quad . \tag{99}$$

From Eqs. (88), (98), and (99) we deduce

$$G_n(s-x) - G_{n+1}(s-x) = \frac{1}{n!}\left(\frac{s-x}{\lambda_A}\right)^n e^{-(s-x)/\lambda_A} \quad , \quad n > 0 \quad ,$$

$$H_n(s-x) - H_{n+1}(s-x) = \frac{1}{n!}\left(\frac{s-x}{\lambda_B}\right)^n e^{-(s-x)/\lambda_B} \quad , \quad n > 0 \quad . \qquad (100)$$

Thus $P[\beta(s) < x]$, given by Eq. (96), becomes

$$P[\beta(s) < x] = e^{-(s-x)/\lambda_A}\left\{1 + \left(\frac{s-x}{\lambda_A\lambda_B}\right)^{1/2} \times \int_0^x dy\, \frac{e^{-y/\lambda_B}}{y^{1/2}}\, I_1\left[2\left(\frac{(s-x)y}{\lambda_A\lambda_B}\right)^{1/2}\right]\right\} \quad , \qquad (101)$$

where we have recognized the Taylor series expansion for the modified Bessel function as

$$I_1(z) = \sum_{r=0}^{\infty} \frac{(z/2)^{2r+1}}{r!(r+1)!} \quad . \qquad (102)$$

A similar result is found for $P[\alpha(s) < x]$. Using the fact that $P[\beta(s) > x] = 1 - P[\beta(s) < x]$ and inserting these results into Eq. (93) gives the cumulative distribution function in the optical depths range $\sigma_B s < \tau < \sigma_A s$ as

$$F(\tau,s) = p_A\left\{1 - e^{-u}\left[1 + 2\int_0^{(uv)^{1/2}} dx I_1(2x)e^{-x^2/u}\right]\right\} + p_B e^{-v}\left[1 + 2\int_0^{(uv)^{1/2}} dx I_1(2x)e^{-x^2/v}\right] \quad ,$$

$$\sigma_B s < \tau < \sigma_A s \quad , \qquad (103)$$

where we have defined

$$u = \frac{1}{\lambda_A}\left(\frac{\tau - \sigma_B s}{\sigma_A - \sigma_B}\right) \quad ; \qquad v = \frac{1}{\lambda_B}\left(\frac{\sigma_A s - \tau}{\sigma_A - \sigma_B}\right) \quad . \tag{104}$$

Equations (94) and (103) give the cumulative distribution function for all physically meaningful (namely positive) values of τ; the geometric distance s is simply a parameter in this distribution. We use this distribution function in the next section to obtain the exact solution to a transmission problem.

IV. SOLUTIONS FOR $\langle\Psi(s)\rangle$ AND NUMERICAL RESULTS

As was stated in the introduction, one way to obtain the solution for $\langle\Psi(s)\rangle$ is to ensemble average the solution $\Psi(s)$ as given by Eq. (3). We carry out the algebraic details of this averaging for the source-free (S=0) problem. In this case Eq. (3) is simply $\Psi(s) = \Psi_o \exp(-\tau)$, and ensemble averaging this pure exponential, we have

$$\langle\Psi(s)\rangle = \Psi_o\langle\exp(-\tau)\rangle = \Psi_o \int_o^\infty d\tau f(\tau,s)e^{-\tau} \quad , \tag{105}$$

where $f(\tau,s)$ is the probability density function for the optical depth random variable τ, with s a parameter in this distribution function. An integration of Eq. (105) by parts introduces the cumulative distribution function $F(\tau,s)$, and we have

$$\langle\Psi(s)\rangle = \Psi_o\left[e^{-\sigma_A s} + \int_{\sigma_B s}^{\sigma_A s} d\tau F(\tau,s)e^{-\tau}\right] \quad , \tag{106}$$

where $F(\tau,s)$ is given by Eq. (103) for our two-fluid Markovian stochastic mixture.

To evaluate the right-hand side of Eq. (106), we introduce the Laplace transform, with a transform variable p, of $\langle\Psi(s)\rangle$ as $\phi(p)$, i.e.,

$$\phi(p) = \int_0^\infty ds e^{-ps}\langle\Psi(s)\rangle \quad . \tag{107}$$

Laplace transforming Eq. (105), we then obtain

$$\phi(p) = \Psi_o\left[(\sigma_A + p)^{-1} + \int_0^\infty ds \int_{\sigma_B s}^{\sigma_A s} d\tau F(\tau,s)e^{-(ps+\tau)}\right] \quad . \tag{108}$$

We change integration variables in Eq. (108) from (s,τ) to (u,v), where u and v are defined by Eq. (104). The double integral in Eq. (105) then becomes a double integral over the first quadrant of (u,v) space. Inserting Eq. (103) for $F(\tau,s)$ we then have

$$\begin{aligned}\phi(p) = {} & \Psi_o(\sigma_A+p)^{-1} + \Psi_o\lambda_A\lambda_B(\sigma_A-\sigma_B)\int_0^\infty du \int_0^\infty dv \\ & \times \left[p_A(1 - e^{-u}) + p_B e^{-v} - 2p_A g(u,v) + 2p_B g(v,u)\right] \\ & \times \exp\{-[\lambda_A(p_A+\sigma_A)u + \lambda_B(p_B+\sigma_B)v]\} \quad , \end{aligned} \tag{109}$$

where we have defined the function $g(u,v)$ as

$$g(u,v) = e^{-u}\int_0^{(uv)^{1/2}} dx I_1(2x)e^{-x^2/u} \quad . \tag{110}$$

The difficult integrations on the right-hand side of Eq. (109) can be written in generic form as

$$I = \int_0^\infty du \int_0^\infty dv e^{-u} e^{-(au+bv)} \int_0^{(uv)^{1/2}} dx I_1(2x) e^{-x^2/u} \quad , \tag{111}$$

where a and b are positive constants. Interchanging the orders of the x and v integrations in Eq. (111) gives

$$I = \int_0^\infty du e^{-(1+a)u} \int_0^\infty dx I_1(2x) e^{-x^2/u} \int_{x^2/u}^\infty dv e^{-bv} \quad . \tag{112}$$

The integral over v is trivial, and if we change integration variables from u to y according to $y = (1+a)u$ we then have

$$I = \frac{1}{b(1+a)} \int_0^\infty dx I_1(2x) \int_0^\infty dy \, \exp\left[-\left(y + \frac{(1+a)(1+b)x^2}{y}\right)\right] \quad . \tag{113}$$

This integral over y can be expressed[13] in terms of the modified Bessel function $K_1(z)$, and we are then left with the single integral over x

$$I = \frac{2}{b}\left(\frac{1+b}{1+a}\right) \int_0^\infty dx x I_1(2x) K_1\{2[(1-a)(1+b)]^{1/2} x\} \quad . \tag{114}$$

This final integral over x can be expressed[13] as a hypergeometric function $F(2,1;\ 2;\ z) = (1-z)^{-1}$, and we obtain the relatively simple result

$$I = \{2b(1+a)[(1+a)(1+b) - 1]\}^{-1} \quad . \tag{115}$$

Using this generic result to integrate the terms involving $g(u,v)$ and $g(v,u)$ in Eq. (109), we obtain after some algebraic simplification,

$$\phi(p) = \Psi_o\left[\frac{p + \tilde{\sigma}}{(p+\tilde{\sigma})(p+\langle\sigma\rangle) - \beta}\right] \quad , \tag{116}$$

where $\langle\sigma\rangle$ is the ensemble averaged cross section given by Eq. (59), β is the coefficient in the two-point auto-correlation function [see Eq. (41)] given by Eq. (69), and $\tilde{\sigma}$ is defined by

$$\tilde{\sigma} = p_B\sigma_A + p_A\sigma_B + \lambda_A^{-1} + \lambda_B^{-1} \quad . \tag{117}$$

Laplace inversion of Eq. (116) then gives the exact result for $\langle\Psi(s)\rangle$ in the source-free, two-fluid Markovian mixture as

$$\langle\Psi(s)\rangle = \Psi_o \left(\frac{r_+ - \tilde{\sigma}}{r_+ - r_-}\right) e^{-r_+ s} + \left(\frac{\tilde{\sigma} - r_-}{r_+ - r_-}\right) e^{-r_- s} \quad , \tag{118}$$

with

$$r_\pm = \frac{1}{2}\left\{\langle\sigma\rangle + \tilde{\sigma} \pm \left[(\langle\sigma\rangle - \tilde{\sigma})^2 + 4\beta\right]^{1/2}\right\} \quad . \tag{119}$$

For equal volume fractions, i.e., $p_A = p_B = 1/2$, Eq. (118) has been obtained earlier by Bourret [14,15] by the method of parastochastic operators in the special case of a dichotomic Markov chain. However, as discussed by Frisch,[4] it is only the $p_A = p_B = 1/2$ result which can be obtained by Bourret's method. This is related to the discussion below Eq. (77) in this paper concerning the vanishing of H_n and K_n for $n > 2$ when $p_A = p_B = 1/2$. Equation (118) has the proper behavior in known limiting cases, namely

$$\langle\Psi(s)\rangle \xrightarrow[\lambda_A \to 0]{} e^{-\sigma_B s} \quad , \tag{120}$$

$$\langle\Psi(s)\rangle \xrightarrow[\lambda_B \to 0]{} e^{-\sigma_A s} \quad , \tag{121}$$

$$\langle\Psi(s)\rangle \xrightarrow[\lambda_A, \lambda_B \to 0]{} e^{-\langle\sigma\rangle s} \quad , \tag{122}$$

$$\langle\Psi(s)\rangle \xrightarrow[\lambda_A,\lambda_B\to\infty]{} p_A\, e^{-\sigma_A s} + p_B\, e^{-\sigma_B s} \quad . \tag{123}$$

There is one additional limit that is interesting to consider, namely $\sigma_B \to 0$ and $\sigma_A \to \infty$. This corresponds physically to the case of packets, with an infinite optical thickness, of fluid A imbedded in a vacuum background. In this case Eq. (118) reduces to

$$\langle\Psi(s)\rangle \xrightarrow[\sigma_A\to\infty]{\sigma_B\to 0} p_B\, e^{-s/\lambda_B} \quad . \tag{124}$$

The factor p_B on the right-hand side of Eq. (124) is just the probability that a particle starts in a packet of fluid B, the vacuum. [If it started in fluid A, it would be absorbed at s=0 since $\sigma_A = \infty$, and hence not contribute to $\langle\Psi(s)\rangle$.] The exponential term in Eq. (124) merely states the correct physical fact that in this limit the mean free path of a particle is just λ_B, the average distance between packets of fluid A.

We can use the exact result for $\langle\Psi(s)\rangle$ given by Eq. (118) to assess the accuracy of the approximate transport models introduced in Sec. II, namely the small fluctuation description given by Eq. (37), and the Nth order Fokker-Planck approximation to this small fluctuation equation, given by Eq. (42). In the source-free (S=0) case with $\langle\sigma\rangle$ independent of position, Eq. (37) is written, using Eq. (41) for the required two-point spatial correlation with η given by Eq. (70),

$$\frac{d\langle\Psi\rangle}{ds} + \langle\sigma\rangle\langle\Psi\rangle = \beta \int_o^s ds_1\, e^{-\hat{\sigma}(s-s_1)} \langle\Psi(s_1)\rangle \quad , \tag{125}$$

with β and $\hat{\sigma}$ given by Eqs. (69) and (77), respectively. The integral in this equation is of the convolution type, and hence Eq. (125) is easily solved by Laplace transforming. The result is

$$\langle\Psi(s)\rangle = \Psi_o\left[\left(\frac{r_+ - \hat{\sigma}}{r_+ - r_-}\right) e^{-r_+ s} + \left(\frac{\hat{\sigma} - r_-}{r_+ - r_-}\right) e^{-r_- s}\right] \quad , \tag{126}$$

where, in this case,

$$r_{\pm} = \frac{1}{2}\left\{\langle\sigma\rangle + \hat{\sigma} \pm [(\langle\sigma\rangle - \hat{\sigma})^2 + 4\beta]^{1/2}\right\} \quad . \tag{127}$$

A comparison of this small fluctuation result [Eqs. (126) and (127)] with the exact result [Eqs. (118) and (119)] shows that they are very similar in form. The only difference is that the small fluctuation result involves $\hat{\sigma}$, whereas the exact result involves $\tilde{\sigma}$ in place of $\hat{\sigma}$. These two results will be identical when $\hat{\sigma} = \tilde{\sigma}$, which occurs for $p_A = p_B = 1/2$; i.e., equal volume fractions of the two fluids. We previously pointed out [see the discussion below Eq. (77)] that the small fluctuation equation is, in fact, exact for all size fluctuations when $p_A = p_B = 1/2$. However, we do not have a physical understanding as to what is special about equal volume fractions for the two fluids which makes the small fluctuation equation exact in this case.

We now consider Eq. (42), the Fokker-Planck approximation to this small fluctuation equation, in the low order cases N=0, 1, and 2. In the source-free ($\langle S\rangle = \alpha = 0$) case, the solution is given by the pure exponential

$$\langle\Psi(s)\rangle = \Psi_o\, e^{-rs} \quad , \tag{128}$$

where the exponent r is given in these three Fokker-Planck approximations by

$$r = \frac{\hat{\sigma}}{2}\left\{\left[\left(\frac{\hat{\sigma}^2}{\beta} + 1\right)^2 + 4\left(\frac{\langle\sigma\rangle\hat{\sigma}}{\beta} - 1\right)\right]^{1/2} - \left(\frac{\hat{\sigma}^2}{\beta} + 1\right)\right\} \;, \quad N = 2 \quad , \tag{129}$$

$$r = \hat{\sigma}\,\frac{\langle\sigma\rangle\hat{\sigma} - \beta}{\hat{\sigma}^2 + \beta} \quad , \quad N = 1 \quad , \tag{130}$$

$$r = (\langle\sigma\rangle\hat{\sigma} - \beta)/\hat{\sigma} \quad , \quad N = 0 \quad . \tag{131}$$

We note that if we neglect the statistical corrections entirely, we have the so-called "atomic mix" approximation, and the transport equation for $\langle\Psi\rangle$ is simply, in the source-free case,

$$\frac{d\langle\Psi\rangle}{ds} + \langle\sigma\rangle\langle\Psi\rangle = 0 \quad . \tag{132}$$

This equation again has Eq. (128) as its solution, with $r = \langle\sigma\rangle$ in this case.

To obtain some idea of the accuracy of these various approximate formulations, we present in Tables 1 and 2 a few typical numerical results. We have set to unity the incident distribution, i.e., $\Psi_o = 1$, and have chosen a length scale such that $\langle\sigma\rangle = 1$ for all cases considered. Also, these two tables give $\langle\Psi(s)\rangle$ at $s = \ln 10$, and hence for all cases the atomic mix result is simply $\langle\Psi(s)\rangle = 0.1$ since $\langle\sigma\rangle s = \ln 10$. The deviation of our exact result for $\langle\Psi\rangle$ as given by Eq. (118) from 0.1 gives an indication of the importance of properly accounting for the statistical nature of the medium in a transport calculation. The deviation of the small fluctuation equation results, and the corresponding Fokker-Planck approximations, from the exact results gives an indication of the accuracy of these various simplified, but approximate, transport descriptions in a random medium.

Table I presents four different cases, each having $\lambda_A = \lambda_B$, and hence $p_A = p_B = 1/2$. As we have already remarked, the small fluctuation equation is exact for all size fluctuations when $p_A = p_B = 1/2$. We see from this table, in particular for the last case, the importance of accounting for the statistical nature of the medium. That is, for this case the atomic mix model which completely ignores this statistical nature underestimates $\langle\Psi\rangle$ by a factor in excess of three. We also see from this table that the lowest order (N=0) Fokker-Planck result is more accurate than the higher order (N=1 and 2) results. This is probably due to extending the lower integration limit in Eq. (37) to $s_1 = -\infty$ in deriving the Fokker-Planck approximation given by Eq. (42). This makes the approximation asymptotic in character; keeping more terms in the sum in Eq. (42) does not necessarily improve the accuracy of the result.

In Table II we present five additional cases, but for these cases $\lambda_A \neq \lambda_B$, and hence $p_A \neq p_B \neq 1/2$. Here we can assess the accuracy of the small fluctuation approximation. We see, from the last two cases in this table, that the small fluctuation model is completely inadequate when the fluctuations are large and the statistical corrections are important (i.e., one is far from the atomic mix limit). In particular, for the last case $\langle\Psi\rangle$ exceeds unity; the small fluctuation equation is predicting growth rather than decay as the particles traverse the medium. This comes about since r_- as given by Eq. (127) is negative. Such growth also occurs for the second to last case in this table, although in this case $\langle\Psi\rangle$ at $s = \ell n\ 10$ is still less than unity. The "complex" entry in this table means that Eq. (129) gave a value for r which is not real.

Based upon these results and other cases we have considered, it appears that the following two conclusions can tentatively be drawn. First, the small fluctuation equation should only be used when the fluctuations are, in fact, small or when $p_A = p_B = 1/2$. Secondly, the Fokker-Planck model, since it is an approximation to the small fluctuation equation, should only be used under the same circumstances, and the N=0 model seems to be the most accurate.

We conclude this section by obtaining an exact solution for $\langle\Psi(s)\rangle$ in the presence of a source ($S\neq 0$) for our two-fluid Markovian mixture. In this case, we solve the exact transport-like equation for $\langle\Psi(s)\rangle$, namely Eq. (21), as opposed to ensemble averaging Eq. (3). The statistical correction terms in Eq. (21), $\hat{T}_n$ and $\tilde{T}_n$, are given by Eqs. (75) and (76), and can be rewritten as

$$L\hat{T}_n = H_n \int_0^s ds_1 R(s-s_1) \int_0^{s_1} ds_2 R(s_1-s_2) \ldots \int_0^{s_{n-2}} ds_{n-1} R(s_{n-2}-s_{n-1}) \quad , \tag{133}$$

$$L\tilde{T}_n = K_n \int_0^s ds_1 R(s-s_1) \int_0^{s_1} ds_2 R(s_1-s_2) \ldots \int_0^{s_{n-2}} ds_{n-1} R(s_{n-2}-s_{n-1}) \langle\Psi(s_{n-1})\rangle, \tag{134}$$

where the kernel $R(s)$ is given by

$$R(s) = \exp(-\hat{\sigma}s) \quad . \tag{135}$$

Written in this way, $L\hat{T}_n$ and $L\tilde{T}_n$ can be seen to be multiple convolution integrals, and hence Eq. (21) can be solved by Laplace transforming. If we again define $\phi(p)$ as the Laplace transform of $\langle\Psi(s)\rangle$ according to Eq. (107), then a Laplace transform of Eq. (21) with $L\hat{T}_n$ and $L\tilde{T}_n$ given by Eqs. (133) and (134) gives

$$\begin{aligned}(p+\langle\sigma\rangle)\phi(p) - \Psi_o + \phi(p) \sum_{n=0}^{\infty} (-1)^{n+1} K_{n+2}(p+\hat{\sigma})^{-(n+1)} \\ - p^{-1} \sum_{n=0}^{\infty} (-1)^{n+1} H_{n+2}(p+\hat{\sigma})^{-(n+1)} = p^{-1}\langle S\rangle \quad .\end{aligned} \tag{136}$$

Using Eqs. (73) and (74) for H_n and K_n in Eq. (136), summing the resulting geometric series, and solving for $\phi(p)$, we find

$$\phi(p) = \frac{\Psi_o(p+\tilde{\sigma}) + p^{-1}[\langle S\rangle(p+\tilde{\sigma})-\alpha]}{(p+\tilde{\sigma})(p+\langle\sigma\rangle) - \beta} \quad , \tag{137}$$

where α, β, and $\tilde{\sigma}$ are given by Eqs. (68), (69), and (117), respectively. The Laplace inversion of Eq. (137) gives the exact result, within the context of our Markov model, for the ensemble averaged distribution function $\langle\Psi(s)\rangle$ as

$$\langle\Psi(s)\rangle = \Psi_o\left[\left(\frac{r_+ - \tilde{\sigma}}{r_+ - r_-}\right) e^{-r_+ s} + \left(\frac{\tilde{\sigma} - r_-}{r_+ - r_-}\right) e^{-r_- s}\right]$$
$$+ \left[\frac{\langle S\rangle(\tilde{\sigma}-r_+) - \alpha}{r_+(r_+-r_-)}\right] e^{-r_+ s} - \left[\frac{\langle S\rangle(\tilde{\sigma}-r_-) - \alpha}{r_-(r_+-r_-)}\right] e^{-r_- s}$$
$$+ \frac{\langle S\rangle\tilde{\sigma} - \alpha}{r_+ r_-} \quad , \qquad (138)$$

with $r_\pm$ again given by Eq. (119). We note that in the absence of a source ($\langle S\rangle=\alpha=0$), Eq. (138) agrees with the result obtained earlier [see Eq. (118)] by ensemble averaging Eq. (3) with S=0. We emphasize that Eq. (138) is an exact expression for $\langle\Psi(s)\rangle$, the ensemble averaged distribution function, but only for time-independent transport through a purely absorbing (no scattering) medium with statistics as described in Sec. III. In particular, this statistical model is a Markovian mixture of two immiscible fluids, and further assumes that the Markov statistical parameters λ_i as well as the fluid parameters σ_i and S_i, i=A,B, are all independent of position.

V. CONCLUDING REMARKS

The work summarized in this paper represents our first attempt at developing a general formalism for describing linear transport through a medium composed of two randomly mixed fluids. We have considered only the very simplest situation, that of time independent transport through a purely absorbing medium for a two-component Markovian mixture, with all parameters λ_i, σ_i, and S_i, i=A,B, independent of position. Clearly, many generalizations suggest themselves. With the inclusion of time dependence and scattering, a generic linear transport equation is given by

$$\frac{1}{v}\frac{\partial\Psi}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}\Psi + (\sigma_a+\sigma_s)\Psi$$

$$= \int_{4\pi} d\vec{\Omega}'\sigma_s(\vec{\Omega}'\to\vec{\Omega})\Psi(\vec{\Omega}') + S \quad , \tag{139}$$

where $\Psi = \Psi(\vec{r},\vec{\Omega},t)$, and the remaining notation in Eq. (139) is standard. In addition to explicit consideration of time dependence and a scattering contribution, one could investigate other (than Markov) statistical models for the random variables σ_a, σ_s, and S.

With regard to the simplified version of Eq. (139) considered in this paper ($\sigma_s = \partial\Psi/\partial t = 0$), an open question is the physical realizability of our Markov model. Along any given direction , one can easily envision a mixture of two types of fluid packets, with each packet of fluid i having an exponential chord length distribution with a mean λ_i. However, can one realize such exponential chord length distributions simultaneously in all directions s in three dimensional geometry? In this regard, we note that if the fluid mixture is composed of alternating fluid slabs, with each slab of fluid i infinite in two dimensions and with an exponentially distributed thickness with mean T_i in the third dimension, then one indeed realizes exponential chord length distributions in all directions s simultaneously. However, the mean chord length will be s dependent and given by $\lambda_i = T_i/\mu$, where μ is the cosine of the angle between the direction s and the normal to the slab surfaces. Can any statement concerning physical realizability be made for non-slab geometry, and can one envision any fluid packet geometry which has exponential chord length distributions with the same mean λ_i in all directions? It would also be interesting to investigate the robustness of the results given in this paper to the statistical model used. Specifically, within the context of a two-fluid mixture, how sensitive are the results to the use of an exponential distribution of chord lengths? We note that the exponential distribution contains only one parameter λ_i, and hence the average chord length (which is just λ_i) and the variance (which is just λ_i^2) are not independent. Hence one might ask how sensitive are our results, e.g., Eq. (138), for given average chord lengths of each fluid component, to the variances (and higher moments) of the chord length distributions? Clearly the applicability of the exponential (or any other) distribution must be established from the underlying physics of the particular transport situation under consideration. We mention parenthetically that the exponential distribution appears to be a fairly good description of the distribution of rock fragment sizes, as discussed by Engleman, Jaeger, and Levi.[16] The hope is that relevant transport results are relatively insensitive to the chord length distributions, thus obviating the need for a detailed chord length description.

We hope to address these points, as well as extensions of our analysis to more general transport equations, in future publications.

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REFERENCES

[1]J. B. Keller, "Wave Propagation in Random Media," in Proc. Symp. Appl. Math., 13, (Amer. Math. Soc., Providence, RI, 1962) p. 227.

[2]J. B. Keller, "Stochastic Equations and Wave Propagation in Random Media," in Proc. Symp. Appl. Math., 16, (Amer. Math. Soc., Providence, RI, 1964) p. 145.

[3]J. B. Keller, "Effective Behavior of Heterogeneous Media," in Statistical Mechanics and Statistical Methods in Theory and Application, edited by U. Landman (Plenum Press, New York, 19 7) p. 632.

[4]U. Frisch, "Wave Propagation in Random Media," in Probabilistic Methods in Applied Mathematics, edited by A. T. Bharucha-Reid (Academic Press, New York, 1968) p. 75.

[5]H. Jacobs, "Influence of Gradual Density Transition and Nonlinear Saturation on Rayleigh-Taylor Instability Growth," Kernforschungszentrum Karlsruhe Report KfK 3756 (Karlsruhe, West Germany, 1984).

[6]H. Frisch and U. Frisch, "LTE and Non-LTE Line Formation with Turbulent Velocity Fields," in Colloques Internationaux du C. N. R. S., 250, Physique des Mouvements dans les Atmospheres Stellaires (Paris, 1975) p. 113.

[7]G. Traving, "Line Formation in Stochastic Media," ibid, p. 145.

[8]E. Parzan, Stochastic Processes (Holden-Day, San Francisco, 1962).

[9]P. Debye and A. M. Bueche, J. Appl. Phys., 20, 518 (1949).

[10]P. Debye, H. R. Anderson, Jr., and H. Brumberger, ibid. 28, 679 (1957).

[11]D. V. Lindley, Introduction to Probability and Statistics from a Bayesian Viewpoint, (Cambridge University Press, Cambridge, 1980), Pt. 1, pp. 246-247.

[12]J. Medhi, Stochastic Processes, (Wiley, New York, 1982).

[13]I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series, and Products, 4th ed. (Academic Press, New York, 1980), pp. 340,693.

[14]R. C. Bourret, Phys. Lett., 12, 323 (1964).

[15]R. C. Bourret, Canad. J. Phys., 44, 2519 (1966).

[16]R. Englman, Z. Jaeger, and A. Levi, Philos. Mag. B, 50, 307 (1984).

TABLE I. $\langle\Psi(\ell n\ 10)\rangle$ for $p_A = p_B = 1/2$.

Parameters	Exact	Fokker-Planck		
		N = 2	N = 1	N = 0
$\lambda_A = 0.1$	0.1001	0.1001	0.1001	0.1001
$\lambda_B = 0.1$				
$\sigma_A = 1.1$				
$\sigma_B = 0.9$				
$\lambda_A = 10.0$	0.1023	0.1049	0.1036	0.1019
$\lambda_B = 10.0$				
$\sigma_A = 1.1$				
$\sigma_B = 0.9$				
$\lambda_A = 0.1$	0.1095	0.1098	0.1097	0.1093
$\lambda_B = 0.1$				
$\sigma_A = 1.9$				
$\sigma_B = 0.1$				
$\lambda_A = 10.0$	0.3592	0.6362	0.6194	0.4732
$\lambda_B = 10.0$				
$\sigma_A = 1.9$				
$\sigma_B = 0.1$				

TABLE II. $\langle\Psi(\ln 10)\rangle$ for $p_A \neq p_B$.

Parameters	Exact	Small Fluctuation Equation	Fokker-Planck N = 2	Fokker-Planck N = 1	Fokker-Planck N = 0
$\lambda_A = 0.02$ $\lambda_B = 0.08$ $\sigma_A = 1.1$ $\sigma_B = 0.975$	0.1000	0.1000	0.1000	0.1000	0.1000
$\lambda_A = 2.0$ $\lambda_B = 8.0$ $\sigma_A = 1.1$ $\sigma_B = 0.975$	0.1004	0.1004	0.1007	0.1006	0.1004
$\lambda_A = 0.02$ $\lambda_B = 0.08$ $\sigma_A = 4.0$ $\sigma_B = 0.25$	0.1083	0.1086	0.1086	0.1086	0.1085
$\lambda_A = 2.0$ $\lambda_B = 8.0$ $\sigma_A = 4.0$ $\sigma_B = 0.25$	0.3694	0.9989	1.6656	1.6131	2.4245
$\lambda_A = 1.0$ $\lambda_B = 9.0$ $\sigma_A = 9.1$ $\sigma_B = 0.1$	0.5802	9.5438	complex	8.5259	283.90